Glueball spectrum based on a rigorous three-dimensional relativistic equation for two-gluon bound states II: calculation of the glueball spectrum

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In the preceding paper, a rigorous three-dimensional relativistic equation for two-gluon bound states was derived from the QCD with massive gluons and represented in the angular momentum representation. In order to apply this equation to calculate the glueball spectrum, in this paper, the equation is recast in an equivalent three-dimensional relativistic equation satisfied by the two-gluon positive energy state amplitude. The interaction Hamiltonian in the equation is exactly derived and expressed as a perturbative series. The first term in the series describes the one-gluon exchange interaction which includes fully the retardation effect in it. This term plus the linear confining potential are chosen to be the interaction Hamiltonian and employed in the practical calculation. With the integrals containing three and four spherical Bessel functions in the QCD vertices being analytically calculated, the interaction Hamiltonian is given an explicit expression in the angular momentum representation. Numerically solving the relativistic equation with taking the contributions arising from the retardation effect and the longitudinal mode of gluon fields into account, a set of masses for the $0^{++}, 0^{-+}, 1^{++}, 1^{-+}, 2^{++}$ and 2^{-+} glueball states are obtained and are in fairly good agreement with the predictions given by the lattice simulation In addition, some new glueball states are predicted.

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I. INTRODUCTION

As mentioned in the preceding paper, searching for glueballs, nowadays, is an challenging task in particle physics. Since there are numerous technical difficulties of giving unambiguous identifications of the glueballs in experiment [1-15], it is expected that the existence of glueballs and their properties could be precisely predicted by theoretical investigations so as to guide the experimental searches. Various methods were proposed in the past to serve such investigations [16-31]. However, the predictions given by different methods are not consistent sometimes and even contradictory with each other [32-33]. Of these methods, the lattice simulation [26-31] is considered to be faithful. Nevertheless, even for this method, there still are controversies on the results given by different calculations. Apart from the lattice computation, the nonrelativistic potential model [16-18], the relativistic Dirac equation [22] and the Bethe-Salpeter (B-S) equation [23-25] have recently been applied to evaluate the glueball spectrum. In these methods, the interaction between gluons is constructed by two parts: the short-range part which is described mainly by the one-gluon exchange interaction and the long-range part which is represented by a phenomenological confining potential. In the nonrelativistic potential model, the short-range interaction is simulated by a potential which is derived in the approximation of order v^2/c^2 where v is the gluon velocity and c the light velocity with the assumption that the gluons in a glueball move not too fast. In a recent work by using this model [18], with the choice of the lightest glueball masses given in the lattice computation [29] as input, the authors obtained a series of two-gluon glueball states with masses below $3 GeV$. Except for some glueball masses which are in pretty good agreement with the lattice predictions, the other calculated masses are apparently different from the lattice results. As was emphasized in Ref.[18], to gain a physical solution to the lightest scalar glueball, it is necessary to additionally introduce a phenomenological smearing function to replace the δ-function in the attractive contact terms of the potential. Otherwise, the Hamiltonian would be unbounded from below. This probably is an unnatural feature of the nonrelativistic potential model. In Ref.[22], the calculation of the glueball spectrum was performed by employing the relativistic Dirac equation and showed only three theoretical results for the lowest glueball states $0^{++}, 2^{++}$ and 3^{++} some of which are not in so good agreement with those given by lattice investigations. In the calculation, a Fermi-Breit potential (the t-channel one-gluon exchange potential) was inserted into the Dirac equation with the assumption that the nature and the force between two gluons are the same as between two quarks. It seems that this assumption ignores the difference between the potential

for quarks and the one for gluons. In addition, it would be mentioned that the Fermi-Breit potential is derived in the nonrelativistic approximation of order v^2/c^2. Therefore, the calculation is not fully relativistic. The relativistic calculation of the glueball spectrum was carried out in the framework of B-S equation [24,25]. Owing to the difficulty of solving a relativistic equation, only a few states were predicted in these calculations. It is noted that in all the previous applications of the B-S equation, the four-dimensional equation was recast in a three-dimensional form in the instantaneous approximation in which the retardation effect is completely neglected. Another point we would like to note here is that in the aforementioned works, the gluons are all viewed as massive. Each of such gluons in general has three degrees of freedom of polarization. Correspondingly, a gluon field should includes three independent spatial components: two transverse fields and one longitudinal field in the three-dimensional space. In this sense, we can say that the Coulomb gauge as taken in Ref.[25] is inappropriate for the massive gluons because in this gauge the longitudinal mode of the field is completely eliminated. Similarly, in Ref.[24], only the transverse gluons are taken into account even though the temporal gauge adopted in the work allows existence of the longitudinal gluons.

In this paper, we intend to investigate the glueball spectrum based on the three-dimensional relativistic equation for two-gluon bound states which was derived in our former paper in the angular momentum representation. This equation is actually a coupled set of equations satisfied by the four B-S amplitudes for a glueball state: one is related to the positive energy states of two gluons, the other three are related to the two gluon states in which there is at least one gluon in the negative energy state. In the next section, we will derive from this coupled equations an equivalent equation obeyed by only the B-S amplitude of the glueball state for which the two gluons are in the positive energy states and give the effective interaction Hamiltonian in the equation a complete form. Since we are unable to compute all the terms in the Hamiltonian at present, we are limited ourself to work in a semi-phenomenological model by which the interaction Hamiltonian in the equation is given by the one-gluon exchange kernel plus the phenomenological linear confining kernel as was usually done in the previous literature [16-18,23-25]. The new aspects of this paper which distinguish from the previous works are: (1) The calculation is fully relativistic and hence includes the contribution arising from all the relativistic effects to the glueball masses; (2) The retardation effect of the one-gluon exchange interaction is completely taken into account; (3) Apart from the transverse modes of the gluon fields, the contribution from the longitudinal mode of the field to the glueball spectrum is appropriately considered; (4) The renormalization effect is considered by the effective QCD coupling constant which was derived in the one-loop approximation and in a mass-dependent subtraction in our previous work [34]. This coupling constant is not only suitable in the high energy domain, but also in the low energy regime; (5) We work in the angular momentum representation. In the this representation, the glueball states are easily constructed. In particular, with completing the radial integrals containing three and four spherical Bessel functions, the gluon vertices are given explicit and analytical expressions which greatly facilitate the numerical task of solving the equation. The theoretical results obtained in this calculation are in quite good agreement with those given in the lattice study [29-31], In addition, some new predictions are presented.

The remainder of this paper is organized as follows. In Section II, we will derive the three-dimensional equation satisfied by the gluon positive energy state B-S amplitude from the coupled equations derived in the preceding paper. In Section III, the interaction Hamiltonian obtained in the tree diagram approximation will be discussed and its explicit expression will be given. Section IV serves to derive the expression of the linear-wise potential which is used to simulate the gluon confinement and incorporated in the glueball equation for numerical calculations. In the last section, the calculated results will be presented and some discussions will be made. In Appendices A and B, the analytical expressions of three-line and four-line gluon vertices are derived respectively.

II. THE THREE-DIMENSIONAL EQUATION FOR THE GLUON POSITIVE ENERGY STATE AMPLITUDE

The three-dimensional equations derived in the preceding paper (which will be called paper I later on) are shown in the following

$$(E_n - \omega_\alpha - \omega_\beta)\chi_{\alpha\beta}(n) = \sum_{\rho\sigma} K(\alpha\beta; \rho\sigma; E_n)\chi_{\rho\sigma}(n), \tag{2.1}$$

where E is the total energy of a glueball state, ω_α and ω_β represent the energies of free gluons 1 and 2 respectively, $\chi_{\alpha\beta}(n)$ stands for the B-S amplitude describing the glueball state which is defined by

$$\chi_{\alpha\beta}(n) = \langle 0^+|\mathbf{a}_\alpha\mathbf{a}_\beta|n\rangle, \tag{2.2}$$

and $K(\alpha\beta; \rho\sigma; E)$ designates the interaction kernel whose closed expression was derived in paper I. In the matrix notation, it is of the form

$$K = \sum_{i,j=1}^{3} \Lambda_j^{(i)}$$

$$= \{\sum_{i=1}^{3} g_i S_i - \sum_{i,j=1}^{3} g_i G_{ij} g_j + \sum_{i,j=1}^{3} g_i G_i G^{-1} G_j g_j\} S^{-1}, \tag{2.3}$$

where the matrices in the above expression were clearly defined in paper I. Noticing the definitions of ω_α and $\mathbf{a}_\alpha$ (see Eqs.(4.9) and (5.16) in paper I),

$$\omega_\alpha = \begin{cases} \omega(k) & \text{if } \xi_\alpha = 1 \\ -\omega(k) & \text{if } \xi_\alpha = -1 \end{cases} \tag{2.4}$$

and

$$\mathbf{a}_\alpha(k) = \begin{cases} \mathbf{a}_\alpha(k) & \text{if } \xi_\alpha = 1 \\ \mathbf{a}_\alpha^+(k) & \text{if } \xi_\alpha = -1 \end{cases}, \tag{2.5}$$

where the subscript α on the right hand side (RHS) of Eq.(2.5) is defined without including ξ_α and hence $\mathbf{a}_\alpha(k)$ and $\mathbf{a}_\alpha^+(k)$ represent the annihilation and creation operators respectively, the equation in Eq.(2.1) may be separately written as

$$\begin{aligned}
[E_n - \omega(k_1) - \omega(k_2)]\chi_{\alpha^+\beta^+}(n) &= \sum_{\rho\sigma} K(\alpha^+\beta^+; \rho\sigma; E_n)\chi_{\rho\sigma}(n), \\
[E_n - \omega(k_1) + \omega(k_2)]\chi_{\alpha^+\beta^-}(n) &= \sum_{\rho\sigma} K(\alpha^+\beta^-; \rho\sigma; E_n)\chi_{\rho\sigma}(n), \\
[E_n + \omega(k_1) - \omega(k_2)]\chi_{\alpha^-\beta^+}(n) &= \sum_{\rho\sigma} K(\alpha^-\beta^+; \rho\sigma; E_n)\chi_{\rho\sigma}(n), \\
[E_n + \omega(k_1) + \omega(k_2)]\chi_{\alpha^-\beta^-}(n) &= \sum_{\rho\sigma} K(\alpha^-\beta^-; \rho\sigma; E_n)\chi_{\rho\sigma}(n),
\end{aligned} \tag{2.6}$$

where the superscripts $\pm$ in $\alpha^\pm$ and $\beta^\pm$ denote $\xi_\alpha, \xi_\beta = \pm 1$, the indices ρ and σ still include the indices ξ_ρ and ξ_σ respectively and

$$\begin{aligned}
\chi_{\alpha^+\beta^+}(n) &= \langle 0^+|\mathbf{a}_\alpha \mathbf{a}_\beta|n\rangle, \chi_{\alpha^+\beta^-}(n) = \langle 0^+|\mathbf{a}_\alpha \mathbf{a}_\beta^+|n\rangle, \\
\chi_{\alpha^-\beta^+}(n) &= \langle 0^+|\mathbf{a}_\alpha^+ \mathbf{a}_\beta|n\rangle, \chi_{\alpha^-\beta^-}(n) = \langle 0^+|\mathbf{a}_\alpha^+ \mathbf{a}_\beta^+|n\rangle.
\end{aligned} \tag{2.7}$$

Following the procedure described in Ref.[35] for fermion systems, the coupled equations in Eq.(2.6) can be reduced to an equivalent equation satisfied by the B-S amplitude $\chi_{\alpha^+\beta^+}(n)$ for the glueball state in which each of gluons is in its positive energy state. For later convenience of derivation, we define

$$\Delta_{ab}(E) = E - a\omega(k_1) - b\omega(k_2), \tag{2.8}$$

where the subscript n in E_n has been suppressed, $a, b = \pm 1$,

$$\begin{aligned}
\phi_{++}(\alpha\beta; E) &= \chi_{\alpha^+\beta^+}(n), \phi_{+-}(\alpha\beta; E) = \chi_{\alpha^+\beta^-}(n), \\
\phi_{-+}(\alpha\beta; E) &= \chi_{\alpha^-\beta^+}(n), \phi_{--}(\alpha\beta; E) = \chi_{\alpha^-\beta^-}(n)
\end{aligned} \tag{2.9}$$

and

$$K_{abcd}(\alpha\beta; \rho\sigma; E) = K(\alpha^a\beta^b; \rho^c\sigma^d; E), \tag{2.10}$$

in which the $\alpha, \beta, \rho, \sigma$ are defined without including the index ξ. With the definitions given in Eqs.(2.8)-(2.10), the equations in Eq.(2.6) can compactly be written as

$$\Delta_{ab}(E)\phi_{ab}(\alpha\beta; E) = \sum_{cd} \sum_{\rho\sigma} K_{abcd}(\alpha\beta; \rho\sigma; E)\phi_{cd}(\rho\sigma; E), \tag{2.11}$$

where $a, b, c, d = \pm 1$. In the product space of momentum k_1, k_2 and angular momentum marked by α, the above equations may be written in the matrix form

$$\Delta_{++}(E)\phi_{++}(E) = K_{++++}(E)\phi_{++}(E) + \sum_{cd \neq ++} K_{++cd}(E)\phi_{cd}(E), \tag{2.12}$$

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$$\Delta_{ab}(E)\phi_{ab}(E) = K_{ab++}(E)\phi_{++}(E) + \sum_{cd\neq++} K_{abcd}(E)\phi_{cd}(E), \tag{2.13}$$

where $ab \neq ++$ and the terms related to $\phi_{++}(E)$ have been separated out from the others. Furthermore, In the space spanned by $\phi_{ab}(E)$ with $ab \neq ++$, we use the matrix representation defined as follows

$$
\begin{aligned}
&\psi(E) = \phi_{++}(E), \phi(E) = \{\phi_{ab}(E)\}, \\
&\Delta_+(E) = \Delta_{++}(E), \Delta(E) = \{\Delta_{ab}(E)\}, \\
&K_+(E) = K_{++++}(E), \overline{K}^t(E) = \{K_{++cd}(E)\}, \\
&\overline{G}(E) = \{K_{ab++}(E)/\Delta_{ab}(E)\}, \}, \\
&G(E) = \{K_{abcd}(E)/\Delta_{ab}(E)\}.
\end{aligned}
\tag{2.14}
$$

According to these definitions, Eqs.(2.12) and (2.13) may be written in the full matrix form

$$\Delta_+(E)\psi(E) = K_+(E)\psi(E) + \overline{K}^t(E)\phi(E), \tag{2.15}$$

$$\phi(E) = \overline{G}(E)\psi(E) + G(E)\phi(E). \tag{2.16}$$

Solving the equation (2.16), we obtain

$$\phi(E) = \frac{1}{1 - G(E)}\overline{G}(E)\psi(E). \tag{2.17}$$

Substituting the above expression into Eq.(2.15), we finally arrive at

$$\Delta_+(E)\psi(E) = V(E)\psi(E), \tag{2.18}$$

where

$$V(E) = K_+(E) + \overline{K}^t(E)\frac{1}{1 - G(E)}\overline{G}(E), \tag{2.19}$$

which is identified itself with the interaction Hamiltonian. Noticing the definition

$$\frac{1}{1 - G(E)} = \sum_{n=0} G^{(n)}(E). \tag{2.20}$$

Eq.(2.19) can be written as

$$V(E) = \sum_{n=0} V^{(n)}(E), \tag{2.21}$$

where

$$
\begin{aligned}
&V^{(0)}(E) = K_+(E), \\
&V^{(1)}(E) = \overline{K}^t(E)\overline{G}(E), \\
&V^{(2)}(E) = \overline{K}^t(E)G(E)\overline{G}(E) \\
&\cdots\cdots.
\end{aligned}
\tag{2.22}
$$

According to the definitions in Eq.(2.14), Eqs.(2.21) and (2.22) may be explicitly written as

$$V(\alpha\beta;\gamma\delta;E) = \sum_{n=0} V^{(n)}(\alpha\beta;\gamma\delta;E), \tag{2.23}$$

where

$$V^{(0)}(\alpha\beta;\gamma\delta;E) = K(\alpha^+\beta^+;\gamma^+\delta^+;E), \tag{2.24}$$

$$V^{(1)}(\alpha\beta;\gamma\delta;E) = \sum_{ab\neq++}\sum_{\rho\sigma} \frac{K(\alpha^+\beta^+;\rho^a\sigma^b;E)K(\rho^a\sigma^b;\gamma^+\delta^+E)}{E - a\omega(k_1) - b\omega(k_2)}, \tag{2.25}$$

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$$V^{(2)}(\alpha\beta;\gamma\delta;E)$$
$$= \sum_{ab\neq++}\sum_{cd\neq++}\sum_{\rho\sigma}\sum_{\mu\nu} \frac{K(\alpha^+\beta^+;\rho^a\sigma^b;E)K(\rho^a\sigma^b;\mu^c\nu^d;E)K(\mu^c\nu^d;\gamma^+\delta^+;E)}{(E-a\omega(k_1)-b\omega(k_2))(E-c\omega(k_1)-d\omega(k_2))} \tag{2.26}$$
$$\cdots\cdots,$$

where we have used the notation shown in Eqs.(2.8) and (2.10). From Eqs.(2.19)-(2.26), it is clear to see that the negative energy states of two gluons act as intermediate states to appear in the effective interaction Hamiltonian and give contributions to the higher order terms in the Hamiltonian. The equation (2.18) written in an explicit form is such that

$$[E - \omega(k_1) - \omega(k_2)]\psi(\alpha\beta;E) = \sum_{\gamma\delta} V(\alpha\beta;\gamma\delta;E)\psi(\gamma\delta;E). \tag{2.27}$$

This is just the equation satisfied by the gluon positive energy state amplitudes for the glueball states in which

$$\psi(\alpha\beta;E) = \chi_{\alpha^+\beta^+}(n). \tag{2.28}$$

III. THE HAMILTONIAN GIVEN IN THE LOWEST ORDER APPROXIMATION

In this section, we plan to discuss the interaction Hamiltonian in the tree diagram approximation of the order of g^2 here g is the QCD coupling constant. This Hamiltonian can only be given by the term shown in Eq.(2.24) because the other terms in the effective Hamiltonian give the contributions which are all higher than g^2. In general, the $K(\alpha^+\beta^+;\gamma^+\delta^+;E)$ should be calculated according to the expression denoted in Eq.(2.3) which includes three parts. The last part in Eq.(2.3) plays the role of cancelling the B-S reducible diagrams contained in the first two parts and gives no contribution of the order g^2. This is because (1) the coefficients $g_1(\alpha\beta;\rho\sigma\lambda)$ and $g_3(\alpha\beta;\rho\sigma\lambda)$ are proportional to g and $g_2(\alpha\beta;\rho\sigma\tau\lambda)$ is proportional to g^2; (2) the Green's functions $G_1(\rho\sigma\lambda;\gamma\delta;t_1-t_2)$ and $G_3(\rho\sigma\lambda;\gamma\delta;t_1-t_2)$ vanish in the lowest order approximation. As for the first part in Eq.(2.3), in the lowest order approximation, it is easy to verify that

$$S_1(\rho\sigma\lambda,\gamma\delta) = \langle 0|[: \mathbf{a}_\rho\mathbf{a}_\sigma : \mathbf{a}_\lambda, \mathbf{a}_\gamma\mathbf{a}_\delta]|0\rangle = 0,$$
$$S_2(\rho\sigma\tau\lambda,\gamma\delta) = \langle 0|[: \mathbf{a}_\rho\mathbf{a}_\sigma\mathbf{a}_\tau : \mathbf{a}_\lambda, \mathbf{a}_\gamma\mathbf{a}_\delta]|0\rangle = 0, \tag{3.1}$$
$$S_3(\rho\sigma\lambda,\gamma\delta) = \langle 0|[: \mathbf{c}_\rho^+\mathbf{c}_\sigma : \mathbf{a}_\lambda, \mathbf{a}_\gamma\mathbf{a}_\delta]|0\rangle = 0,$$

where $|0\rangle$ denotes the bare vacuum state. Therefore, we only need to consider the second part in Eq.(2.3). In this part, the term related to $g_2(\alpha\beta;\rho\sigma\tau\lambda)$ gives the contribution of order g^4 in the lowest approximation and hence is beyond our consideration. For the terms associated with $g_3(\alpha\beta;\rho\sigma\lambda)$,the relevant Green's functions vanish in the lowest order approximation. Thus, we are only left with terms in the second part of Eq.(2.3) such that

$$K^0(\alpha^+\beta^+;\gamma^+\delta^+;E) = \sum_{\rho\sigma} \Lambda(\alpha^+\beta^+;\rho\sigma;E)S^{-1}(\rho\sigma;\gamma^+\delta^+), \tag{3.2}$$

where

$$\Lambda(\alpha^+\beta^+;\rho\sigma;E)$$
$$= -\sum_{\xi\eta\lambda}\sum_{\mu\nu\tau} g_1(\alpha^+\beta^+;\xi\eta\lambda)G_{11}(\xi\eta\lambda;\mu\nu\tau;E)g_1(\mu\nu\tau;\rho\sigma) \tag{3.3}$$

and the indices $\rho,\sigma,\cdots$ should be understood as $\rho^\pm,\sigma^\pm,\cdots$.

Let us first compute the inverse $S^{-1}(\rho\sigma;\gamma^+\delta^+)$. For this purpose, we operate on the both sides of Eq.(3.2) with S from the right and get

$$\Lambda(\alpha^+\beta^+;\rho\sigma;E) = \sum_{\gamma\delta} K^0(\alpha^+\beta^+;\gamma^+\delta^+;E)S(\gamma^+\delta^+;\rho\sigma). \tag{3.4}$$

It is easy to verify that except for $S(\gamma^+\delta^+;\rho^-\sigma^-)$, the $S(\gamma^+\delta^+;\rho^-\sigma^+)$, $S(\gamma^+\delta^+;\rho^+\sigma^-)$ and $S(\gamma^+\delta^+;\rho^+\sigma^+)$ are all vanishing in the lowest order approximation. As for the $S(\gamma^+\delta^+;\rho^-\sigma^-)$, we have

$$S(\gamma^+\delta^+;\rho^-\sigma^-) = \delta_{\gamma\rho}\delta_{\delta\sigma} + \delta_{\gamma\sigma}\delta_{\delta\rho}. \tag{3.5}$$

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Substituting Eq.(3.5) in Eq.(3.4), we find

$$\Lambda(\alpha^+\beta^+;\rho^-\sigma^-;E) = K^0(\alpha^+\beta^+;\rho^+\sigma^+;E) + K^0(\alpha^+\beta^+;\sigma^+\rho^+;E). \tag{3.6}$$

Since we may interchange the indices ρ and σ in Eq.(2.1) or in Eq.(2.27), noticing $\chi_{\rho\sigma}(n) = \chi_{\sigma\rho}(n)$ or $\psi(\rho\sigma;E) = \psi(\sigma\rho;E)$, we can write from Eq.(3.6) the following relation

$$K^0(\alpha^+\beta^+;\rho^+\sigma^+;E) = \frac{1}{2}\Lambda(\alpha^+\beta^+;\rho^-\sigma^-;E), \tag{3.7}$$

which means that we may set

$$S^{-1}(\rho\sigma;\gamma^+\delta^+) = S^{-1}(\rho^-\sigma^-;\gamma^+\delta^+) = \frac{1}{2}\delta_{\gamma\rho}\delta_{\delta\sigma}. \tag{3.8}$$

Combining Eqs.(3.3) and (3.7), we have

$$\begin{aligned}&K^0(\alpha^+\beta^+;\rho^+\sigma^+;E)\\&= -\tfrac{1}{2}\sum_{\gamma\delta\lambda}\sum_{\mu\nu\tau} g_1(\alpha^+\beta^+;\gamma\delta\lambda)G_{11}(\gamma\delta\lambda;\mu\nu\tau;E)g_1(\mu\nu\tau;\rho^-\sigma^-).\end{aligned} \tag{3.9}$$

In accordance with the definition of $g_1(\alpha\beta;\gamma\delta\lambda)$ and $g_1(\mu\nu\tau;\rho\sigma)$ (see Eqs.(5.7), (5.9), (5.10) and (6.4) in paper I), we can write

$$\begin{aligned}g_1(\alpha^+\beta^+;\gamma\delta\lambda) &= \sum_{\tau} f_1(\gamma\delta\tau)\Delta_{\alpha^+\beta^+;\tau\lambda},\\ g_1(\mu\nu\tau;\rho^-\sigma^-) &= \sum_{\lambda} f_1(\mu\nu\lambda)\Delta_{\rho^-\sigma^-;\lambda\tau},\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}\Delta_{\alpha^+\beta^+;\tau\lambda} &= \Delta_{\alpha^+\tau}\delta_{\beta^+\lambda} + \Delta_{\beta^+\tau}\delta_{\alpha^+\lambda},\\ \Delta_{\rho^-\sigma^-;\lambda\tau} &= \Delta_{\rho^-\lambda}\delta_{\sigma^-\tau} + \Delta_{\sigma^-\lambda}\delta_{\rho^-\tau}\end{aligned} \tag{3.11}$$

and

$$f_1(\alpha\beta\gamma) = A(\alpha\beta\gamma) + A(\alpha\gamma\beta) + A(\gamma\alpha\beta), \tag{3.12}$$

here $A(\alpha\beta\gamma)$ is the three-line gluon vertex given in the angular momentum representation. Considering the expressions in Eq.(3.11) and the fact that only $\Delta_{\alpha^+\beta^-} = -\Delta_{\alpha^-\beta^+} = \delta_{\alpha\beta}$ are nonvanishing for $\Delta_{\alpha\beta}$, Eq.(3.10) can be represented as

$$\begin{aligned}g_1(\alpha^+\beta^+;\gamma\delta\lambda) &= f_1(\gamma\delta\alpha^-)\delta_{\beta^+\lambda} + f_1(\gamma\delta\beta^-)\delta_{\alpha^+\lambda},\\ g_1(\mu\nu\tau;\rho^-\sigma^-) &= -f_1(\mu\nu\rho^+)\delta_{\sigma^-\tau} - f_1(\mu\nu\sigma^+)\delta_{\rho^-\tau}.\end{aligned} \tag{3.13}$$

On inserting Eq.(3.13) into Eq.(3.9), one gets

$$\begin{aligned}K^0(\alpha^+\beta^+;\rho^+\sigma^+;E) &= \tfrac{1}{2}[\Lambda(\alpha^+\beta^+;\rho^+\sigma^+;E) + (\alpha\leftrightarrow\beta)\\ &\quad+(\rho\leftrightarrow\sigma) + (\alpha\leftrightarrow\beta,\rho\leftrightarrow\sigma)],\end{aligned} \tag{3.14}$$

where

$$\Lambda(\alpha^+\beta^+;\rho^+\sigma^+;E) = \sum_{\gamma\delta,\mu\nu} f_1(\gamma\delta\alpha^-)G_{11}(\gamma\delta\beta^+;\mu\nu\sigma^-;E)f_1(\mu\nu\rho^+) \tag{3.15}$$

and the other terms in Eq.(3.14) can be obtained from the first term by exchanging the indices as shown in Eq.(3.14). Now, let us calculate the Green's functions $G_{11}(\gamma\delta\beta^+;\mu\nu\sigma^-;E)$ in the lowest order approximation which are the Fourier transform of the Green functions $G_{11}(\gamma\delta\beta^+;\mu\nu\sigma^-;t_1 - t_2)$. With the aid of Wick theorem, it can be found that only the following Green's function is nonvanishing

$$\begin{aligned}&G_{11}(\gamma^+\delta^+\beta^+;\mu^-\nu^-\sigma^-;t_1 - t_2)\\&= \langle 0|T\{: \mathbf{a}_\gamma(t_1)\mathbf{a}_\delta(t_1):\mathbf{a}_\beta(t_1):\mathbf{a}_\mu^+(t_2)\mathbf{a}_\nu^+(t_2):\mathbf{a}_\sigma^+(t_2)\}|0\rangle,\end{aligned} \tag{3.16}$$

where $\mathbf{a}_\gamma(t_1)$ and $\mathbf{a}_\sigma^+(t_2)$ are the annihilation and creation operators in the interaction picture. Noticing

$$\mathbf{a}_\gamma(t_1) = \mathbf{a}_\gamma e^{-i\omega_\gamma t_1}, \mathbf{a}_\sigma^+(t_2) = \mathbf{a}_\sigma^+ e^{i\omega_\sigma t_2} \tag{3.17}$$

and applying the Wick theorem, we find

$$
\begin{aligned}
&G_{11}(\gamma^+\delta^+\beta^+;\mu^-\nu^-\sigma^-;t_1-t_2) \\
&= \theta(t_1-t_2)e^{-i(\omega_\gamma+\omega_\delta+\omega_\beta)(t_1-t_2)}[\delta_{\gamma\mu}\delta_{\delta\nu}\delta_{\beta\sigma} + \delta_{\gamma\nu}\delta_{\delta\mu}\delta_{\beta\sigma} \\
&+\delta_{\gamma\sigma}\delta_{\delta\mu}\delta_{\beta\nu} + \delta_{\gamma\sigma}\delta_{\delta\nu}\delta_{\beta\mu} + \delta_{\gamma\mu}\delta_{\delta\sigma}\delta_{\beta\nu} + \delta_{\gamma\nu}\delta_{\delta\sigma}\delta_{\beta\mu}].
\end{aligned}
\tag{3.18}
$$

By the Fourier transformation and using the familiar integral representation of the step function, we obtain

$$
\begin{aligned}
&G_{11}(\gamma^+\delta^+\beta^+;\mu^-\nu^-\sigma^-;E) \\
&= \frac{1}{i}\int_{-\infty}^{+\infty} dt e^{iEt} G_{11}(\gamma^+\delta^+\beta^+;\mu^-\nu^-\sigma^-;t) \\
&= \frac{1}{E-\omega_\gamma-\omega_\delta-\omega_\beta+i\epsilon}[\delta_{\gamma\mu}\delta_{\delta\nu}\delta_{\beta\sigma} + \delta_{\gamma\nu}\delta_{\delta\mu}\delta_{\beta\sigma} + \delta_{\gamma\sigma}\delta_{\delta\mu}\delta_{\beta\nu} \\
&+\delta_{\gamma\sigma}\delta_{\delta\nu}\delta_{\beta\mu} + \delta_{\gamma\mu}\delta_{\delta\sigma}\delta_{\beta\nu} + \delta_{\gamma\nu}\delta_{\delta\sigma}\delta_{\beta\mu}].
\end{aligned}
\tag{3.19}
$$

Substituting the above expression in Eq.(3.15), we are led to

$$
\begin{aligned}
&\Lambda(\alpha^+\beta^+;\rho^+\sigma^+;E) \\
&= \sum_{\gamma\delta} \frac{1}{E_n-\omega_\gamma-\omega_\delta-\omega_\beta+i\epsilon} f_1(\gamma^+\delta^+\alpha^-)\{[f_1(\gamma^-\delta^-\rho^+) \\
&+f_1(\delta^-\gamma^-\rho^+)]\delta_{\beta\sigma} + [f_1(\delta^-\beta^-\rho^+) + f_1(\beta^-\delta^-\rho^+)]\delta_{\gamma\sigma} \\
&+[f_1(\gamma^-\beta^-\rho^+) + f_1(\beta^-\gamma^-\rho^+)]\delta_{\delta\sigma}\}.
\end{aligned}
\tag{3.20}
$$

Observing the above expression of $\Lambda(\alpha^+\beta^+;\rho^+\sigma^+;E)$, we see, the first term containing $\delta_{\beta\sigma}$ is an unconnected term. It gives the one-loop correction to the gluon propagator whose effect will be included in the QCD effective coupling constant by the renormalization procedure. The second term proportional to $\delta_{\gamma\sigma}$ describes the one-gluon exchange interaction between two gluons. The third term is the exchanged term for the one-gluon exchange interaction. By dropping the first term, we have

$$
\begin{aligned}
&\Lambda(\alpha^+\beta^+;\rho^+\sigma^+;E) \\
&= \sum_\tau \frac{[f_1(\tau^+\sigma^+\alpha^-) + f_1(\sigma^+\tau^+\alpha^-)][f_1(\tau^-\beta^-\rho^+) + f_1(\beta^-\tau^-\rho^+)]}{E-\omega_\tau-\omega_\sigma-\omega_\beta+i\epsilon},
\end{aligned}
\tag{3.21}
$$

where the summation over τ is performed with respect to the gluon intermediate states and the function f_1 was represented in Eq.(3.12) in terms of the function $A(\alpha\beta\gamma)$ whose explicit expression is derived in Appendix A and shown in the following.

$$
\begin{aligned}
&A(\alpha_1\alpha_2\alpha_3) = -\frac{g}{2}\left(\frac{2}{\pi}\right)^{\frac{3}{2}} f^{abc} k_3 \prod_{i=1}^3 \frac{1}{\sqrt{2\omega(k_i)}} k_i B^{\xi_i}(l_i)_{\lambda_i\tau_i} T_{\lambda_3\lambda_j} \\
&\times J_{l_1'l_2'l_3'}(k_1,k_2,k_3)\Gamma(l_i,l_i',m_i,\xi_i),
\end{aligned}
\tag{3.22}
$$

where

$$
\begin{aligned}
&J_{l_1'l_2'l_3'}(k_1,k_2,k_3) = \frac{\pi^{\frac{5}{2}}}{4}(-1)^{\frac{1}{2}(l_1'+l_2'+l_3')} \\
&\times \sum_{\mu_1,\mu_2,\mu_3=0}^{\infty} \delta_{2(\mu_1+\mu_2+\mu_3),l_1'+l_2'+l_3'} \prod_{i=1}^3 \frac{k_i^{2\mu_i-l_i'-1}}{\Gamma(\mu_i+1)\Gamma(\mu_i-l_i'+\frac{1}{2})},
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&\Gamma(l_i,l_i',m_i,\eta_i) \\
&= \frac{i}{\sqrt{2\pi}} \prod_{i=1}^3 (-1)^{(l_i+l_i'+m_i+1)\sin[\frac{(1-\eta_i)\pi}{4}]}[(2l_i+1)(2l_i'+1)]^{\frac{1}{2}} \\
&\times \begin{pmatrix} l_1' & l_2' & l_3' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ l_1' & l_2' & l_3' \\ l_1 & l_2 & l_3 \end{Bmatrix},
\end{aligned}
\tag{3.24}
$$

$T_{\lambda_3\lambda_j}$ and $B^{\xi_i}(l_i)_{\lambda_i\tau_i}$ are defined respectively in (A.5) and (A.15). It is noted that for a given set of angular momenta, due to the restriction of $\delta_{2(\mu_1+\mu_2+\mu_3),l_1'+l_2'+l_3'}$, only a few terms in the series of Eq.(3.23) survive.

7

IV. THE GLUEBALL EQUATION WITH INCLUSION OF THE CONFINEMENT

As shown in Sec.II, the interaction Hamiltonian in the exact relativistic equation for the glueball states is much complicated. In practical investigations, usually, one only considers the lowest order term in the Hamiltonian which was explicitly derived in the angular momentum representation in the preceding section. Obviously, in order to get reasonable theoretical results, it is necessary to introduce a certain confining potential to phenomenologically simulate all the higher order terms in the Hamiltonian [16-18, 23-25]. How to determine the form of confining interaction in the relativistic equation ? In this paper, as was similarly done for the quark-antiquark system [36], we introduce the confining Hamiltonian operator in such a manner

$$
\begin{aligned}
&H_c \\
&= \frac{1}{2} \int d^3x_1 d^3x_2 f^{abe} f^{cde} \overrightarrow{A^{\vec{a}}}(\overrightarrow{x}_1) \cdot \overrightarrow{A^{\vec{b}}}(\overrightarrow{x}_1) V(|\overrightarrow{x}_1 - \overrightarrow{x}_2|) \overrightarrow{A^{\vec{c}}}(\overrightarrow{x}_2) \cdot \overrightarrow{A^{\vec{d}}}(\overrightarrow{x}_2),
\end{aligned}
\tag{4.1}
$$

where $\overrightarrow{A^{\vec{a}}}(\overrightarrow{x})$ stands for the gluon field operator and $V(|\overrightarrow{x}_1 - \overrightarrow{x}_2|)$ the confining potential which will be taken to be the linear one [37]

$$
V(|\overrightarrow{x}_1 - \overrightarrow{x}_2|) = \gamma |\overrightarrow{x}_1 - \overrightarrow{x}_2|,
\tag{4.2}
$$

here the parameter γ designates the strength of the confining potential. When the gluon field operators in Eq.(4.1) are replaced by their expansions in terms of the multipole fields (see the expansion given in Eq.(4.14) in paper I), Eq.(4.1) will be represented as

$$
H_c = \sum_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} V_c(\alpha_1 \alpha_3; \alpha_2 \alpha_4) : \mathbf{a}_{\alpha_1} \mathbf{a}_{\alpha_2} \mathbf{a}_{\alpha_3} \mathbf{a}_{\alpha_4} :,
\tag{4.3}
$$

where

$$
\begin{aligned}
&V_c(\alpha_1 \alpha_3; \alpha_2 \alpha_4) \\
&= \frac{1}{2} f^{abe} f^{cde} \int d^3x_1 d^3x_2 \overrightarrow{A}_{\beta_1}^{\lambda_1}(\overrightarrow{x}_1) \cdot \overrightarrow{A}_{\beta_2}^{\lambda_2}(\overrightarrow{x}_1) V(|\overrightarrow{x}_1 - \overrightarrow{x}_2|) \overrightarrow{A}_{\beta_3}^{\lambda_3}(\overrightarrow{x}_2) \cdot \overrightarrow{A}_{\beta_4}^{\lambda_4}(\overrightarrow{x}_2).
\end{aligned}
\tag{4.4}
$$

Here the symbols λ_i and β_i were defined in Appendix A. This is just the wanted confining potential written in the angular momentum representation which will be inserted into the relativistic equation. By using the Fourier transformation

$$
|\overrightarrow{x}_1 - \overrightarrow{x}_2| = -\int \frac{d^3q}{(2\pi)^3} \frac{8\pi}{\overrightarrow{q}^4} e^{i\overrightarrow{q} \cdot (\overrightarrow{x}_1 - \overrightarrow{x}_2)},
\tag{4.5}
$$

the expansion for the plane wave function

$$
e^{\overrightarrow{p} \cdot \overrightarrow{x}} = 4\pi \sum_{lm} i^l j_l(pr) Y_{lm}^*(\widehat{p}) Y_{lm}(\widehat{x})
\tag{4.6}
$$

and the expression for the scalar multipole field

$$
A_{lm}^S(k\overrightarrow{x}) = \sqrt{\frac{2}{\pi}} k j_l(kr) Y_{lm}(\widehat{x}),
\tag{4.7}
$$

we can write

$$
V(|\overrightarrow{x}_1 - \overrightarrow{x}_2|) = -8\pi\gamma \sum_{lm} \int_0^\infty dq \frac{1}{q^4} A_{lm}^S(q\overrightarrow{x}_1) A_{lm}^{S*}(q\overrightarrow{x}_2).
\tag{4.8}
$$

Substitution of this expression into Eq.(4.4) leads to

$$
V_c(\alpha_1 \alpha_3; \alpha_2 \alpha_4) = -4\pi\gamma f^{abe} f^{cde} \sum_{lm} \int_0^\infty dq \frac{1}{q^4} D_{\alpha_1 \alpha_2 \tau} D_{\alpha_3 \alpha_4 \tau}^*,
\tag{4.9}
$$

where

$$D_{\alpha_1\alpha_2\tau} = \int d^3x_1 \overrightarrow{A}^{\lambda_1}_{\beta_1}(\overrightarrow{x}_1) \cdot \overrightarrow{A}^{\lambda_2}_{\beta_2}(\overrightarrow{x}_1) A^S_\tau(\overrightarrow{x}_1),$$
$$D_{\alpha_3\alpha_4\tau} = \int d^3x_2 \overrightarrow{A}^{\lambda_3}_{\beta_3}(\overrightarrow{x}_2) \cdot \overrightarrow{A}^{\lambda_4}_{\beta_4}(\overrightarrow{x}_2) A^{S*}_\tau(\overrightarrow{x}_2), \tag{4.10}$$

here $\tau = (q, l, m)$.

Completely analogous to the calculations described in Appendix A, it is easy to derive the following expression

$$V_c(\alpha_1\alpha_3; \alpha_2\alpha_4)$$
$$= -\frac{32}{\pi^2} \lambda f^{abe} f^{cde} \sum_{lm} \int_0^\infty dq \frac{1}{q^2} \prod_{i=1}^4 k_i B^{\xi_i}(l_i)_{\lambda_i\tau_i}$$
$$\times J_{l_1'l_2'l}(k_1, k_2, q) J_{l_3'l_4'l}(k_3, k_4, q) \tag{4.11}$$
$$\times \widehat{\Gamma}(l_i, l_i', m_i, \eta_i, l, m, 1)\widehat{\Gamma}(l_j, l_j', m_j, \eta_j, l, m, -1),$$

where $i = 1, 2$, $j = 3, 4$,

$$\widehat{\Gamma}(l_i, l_i', m_i, \eta_i, l, m, \eta)$$
$$\equiv \int d\Omega(\widehat{x}) \overrightarrow{Y}^{\eta_1}_{l_1l_1'm_1}(\widehat{x}) \cdot \overrightarrow{Y}^{\eta_2}_{l_2l_2'm_2}(\widehat{x}) Y^{\eta}_{lm}(\widehat{x})$$
$$= \prod_{i=1}^2 (-1)^{(l_i+l_i'+m_i+1)\sin[\frac{(1-\eta_i)\pi}{4}]}(-1)^{m\sin[\frac{(1-\eta)\pi}{4}]} \tag{4.12}$$
$$\times \widetilde{\Gamma}(l_i, l_i', \eta_i m_i, l, \eta m)$$

with

$$\widetilde{\Gamma}(l_i, l_i', m_i, l, m) \equiv \int d\Omega(\widehat{x}) \overrightarrow{Y}_{l_1l_1'm_1}(\widehat{x}) \cdot \overrightarrow{Y}_{l_2l_2'm_2}(\widehat{x}) Y_{lm}(\widehat{x})$$
$$= (-1)^{l_1+l_2'+l} [\frac{(2l_1+1)(2l_2+1)(2l_1'+1)(2l_2'+1)(2l+1)}{4\pi}]^{\frac{1}{2}} \tag{4.13}$$
$$\times \begin{pmatrix} l_1' & l_2' & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l \\ l_2' & l_1' & 1 \end{Bmatrix},$$

and $J_{l_il_j'l}(k_i, k_j, q)$ is defined as the same as given in Eq.(3.23). With the introduction of the above confining potential, the total interaction Hamiltonian in Eq.(2.27) is now taken to be

$$V(\alpha\beta; \gamma\delta) = V_g(\alpha\beta; \gamma\delta) + V_c(\alpha\beta; \gamma\delta), \tag{4.14}$$

where

$$V_g(\alpha\beta; \gamma\delta) = K^0(\alpha\beta; \gamma\delta), \tag{4.15}$$

which was formulated in Eqs.(3.9)-(3.24) and the $V_c(\alpha\beta; \gamma\delta)$ was given in Eqs.(4.11)-(4.13).

Now we turn to discuss the wave function $\psi(\alpha\beta)$ in Eq.(2.27) which was defined in Eq.(2.28). In the lowest order approximation, the two-gluon bound states can be written in the form

$$|n\rangle = \sum_{\alpha\beta} f^n_{\alpha\beta} \mathbf{a}^+_\alpha \mathbf{a}^+_\beta |0\rangle, \tag{4.16}$$

where

$$f^n_{\alpha\beta} = \frac{1}{\sqrt{8}} \delta_{c_1c_2} C^{JM}_{l_1m_1l_2m_2} f^{J\pi}_{\lambda_1l_1, \lambda_2l_2}(k_1, k_2), \tag{4.17}$$

in which $\delta_{c_1c_2}$ represents the color singlet, $C^{JM}_{l_1m_1l_2m_2}$ is the C-G coupling coefficient, $\alpha = (c_1, \lambda_1, l_1, m_1, k_1, \xi_\alpha = +1)$, $\beta = (c_2, \lambda_2, l_2, m_2, k_2, \xi_\beta = +1)$, J, M are the total angular momentum and its third component of a glueball and π denotes the spatial parity and charge conjugation parity. With the introduction of the cluster coordinates

$$\overrightarrow{K} = \overrightarrow{k}_1 + \overrightarrow{k}_2, \qquad \overrightarrow{k} = \frac{1}{2}(\overrightarrow{k}_1 - \overrightarrow{k}_2), \tag{4.18}$$

where $\overrightarrow{K}$ and $\overrightarrow{k}$ are the total momentum and relative momentum respectively, we see, in the center of mass system ($\overrightarrow{K} = 0$), Eq.(4.17) reads

$$f^n_{\alpha\beta} = \frac{1}{\sqrt{8}} \delta_{c_1c_2} C^{JM}_{l_1m_1l_2m_2} f^{J\pi}_{\lambda_1l_1, \lambda_2l_2}(k)\delta(k_1 - k_2), \tag{4.19}$$

where $k = |\vec{k}| = k_1 = k_2$. Substituting Eq.(4.16) into Eq.(2.2), we find

$$\psi(\alpha\beta) = \chi_{\alpha^+\beta^+}(n) = \frac{1}{\sqrt{8}}\delta_{c_1 c_2}C^{JM}_{l_1 m_1 l_2 m_2}g^{J\pi h}_{\lambda_1 l_1, \lambda_2 l_2}(k)\delta(k_1 - k_2), \tag{4.20}$$

where

$$g^{J\pi h}_{\lambda_1 l_1, \lambda_2 l_2}(k) = f^{J\pi}_{\lambda_1 l_1, \lambda_2 l_2} + (-1)^h f^{J\pi}_{\lambda_2 l_2, \lambda_1 l_1}, \tag{4.21}$$

in which $h = l_1 + l_2 - J$. Evidently, if $\lambda_1 = \lambda_2, l_1 = l_2$ and $h = odd,$ we have $g^{J\pi h}_{\lambda_1 l_1, \lambda_2 l_2}(k) = 0$. This gives a new selection rule for the glueball states. Upon substituting Eq.(4.20) into Eq.(2.27) and noticing

$$\begin{aligned}&\alpha = (c, \lambda_1, l_1, m_1, k, +1), \beta = (c, \lambda_2, l_2, m_2, k, +1),\\&\rho = (c', \lambda_3, l_3, m_3, q, +1), \sigma = (c', \lambda_4, l_4, m_4, q, +1)\end{aligned} \tag{4.22}$$

and

$$\sum_\alpha \equiv \sum_{c\lambda lm}\int_0^\infty dk, \tag{4.23}$$

we finally arrive at

$$\begin{aligned}&(E - 2\omega)g^{J\pi h}_{\lambda_1 l_1, \lambda_2 l_2}(k)\\&= \frac{1}{8(2J+1)}\sum_{\lambda_3 l_3}\int_0^\infty dq V(\lambda_1 l_1 k; \lambda_2 l_2 k; \lambda_3 l_3 q; \lambda_4 l_4 q; E)g^{J\pi h}_{\lambda_3 l_3, \lambda_4 l_4}(q),\end{aligned} \tag{4.24}$$

where $\omega = \sqrt{k^2 + \mu^2}$, E is the mass of a glueball state given in the center of mass frame and

$$\begin{aligned}&V(\lambda_1 l_1 k; \lambda_2 l_2 k; \lambda_3 l_3 q; \lambda_4 l_4 q; E)\\&= \sum_{m_1 m_3 m_2 m_4 cc' M} V(\alpha\beta; \rho\sigma; E)C^{JM}_{l_1 m_1 l_2 m_2}C^{JM}_{l_3 m_3 l_4 m_4},\end{aligned} \tag{4.25}$$

here $V(\alpha\beta; \rho\sigma; E)$ was given in Eq.(4.14). When the explicit expression of $V(\alpha\beta; \rho\sigma; E)$ is substituted in Eq.(4.25), one can see that the summation over m_1, m_2, m_3, m_4 and M is easily carried out by utilizing the well-known formula for the angular momentum coupling and the summation over the color indices c and c' can be completed by noticing $f^{abc}f^{abc} = 24$. We think, it is unnecessary to show here the result given by these summations.

The equation in Eq.(4.24) is the eigenvalue equation used to calculate the glueball spectrum. In the calculation, the QCD coupling constant g contained in the part of Hamiltonian $V_g(\alpha\beta; \gamma\delta)$ is replaced by the running one which was derived in Ref.[34] recently in the one-loop approximation and in a mass-dependent momentum space subtraction. The coupling constant used in this calculation is of the form

$$\alpha_s(\lambda) = \frac{\alpha_s^0}{1 + \frac{\alpha_s^0}{2\pi}G(\lambda)}, \tag{4.26}$$

where $\alpha_s(\lambda) = g^2(\lambda)/4\pi$,

$$G(\lambda) = 11\ln\lambda - \frac{2}{3}N_f[2 + \sqrt{3}\pi - \frac{2}{\lambda^2} + (1 + \frac{2}{\lambda^2})\frac{1}{\lambda}\eta(\lambda)], \tag{4.27}$$

in which N_f is the number of quark flavors,

$$\eta(\lambda) = \sqrt{\lambda^2 - 4}\ln\frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \tag{4.28}$$

and $\lambda = \sqrt{\frac{p^2 + \mu^2}{\Lambda_{QCD}^2}}$ here p is chosen to be the transfer momentum of the exchanged gluon which may simply be taken as $p = k - q$ for simplicity and Λ_{QCD} is the QCD scale parameter. The running coupling constant shown above is applicable not only in the high energy domain, but also in the low energy regime. Particularly, in the large momentum limit, it immediately goes over to the result obtained previously in the minimal subtraction scheme.

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V. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we first show the theoretical glueball masses calculated from the equation given in Eq.(4.24) and then make some discussions. Our calculation is performed by using the standard program of Mathematica which allows us to compute the effective Hamiltonian in Eq.(4.24) analytically. In this paper, we confine ourself to investigate the low-lying glueball states including the $0^{++}, 0^{-+}, 1^{++}, 1^{-+}, 2^{++}$ and 2^{-+} ground and lower excited states whose masses are less than $4.0 GeV$. Some of these states have been investigated before in various models. We also examine the effects of the longitudinal mode of the multipole fields and the different sets of free parameters on the glueball masses. In our calculation, the theoretical parameters are adjusted so as to be able to compare our results to those presented recently by the lattice simulation [29]. The parameters taken are: the gluon mass $\mu = 0.42 GeV$ which is comparable with $\mu = (0.5 \pm 0.2) GeV$ taken previously in the nonperturbative continuum studies[38], the scale parameter $\Lambda_{QCD} = 0.45 GeV$ and the strength of the confining potential $\gamma = 0.18 GeV^2$, which satisfies the relation $\mu \sim \Lambda_{QCD}$ and $\gamma \sim \Lambda_{QCD}^2$ which make the parameters essentially depend on a single dimensional quantity. Moreover, the value of $\gamma = 0.18 GeV^2$ is consistent with that of the string tension in lattice simulations. The coupling constant $\alpha_s^0 = 0.3$ and quark flavor $N_f = 3$. The calculated masses of glueball states are displayed in table I. In the table, the case I and the case II respectively denote the results obtained with and without considering the contribution arising from the longitudinal mode of the multipole fields which appears in the intermediate states of the matrix elements of the interaction Hamiltonian. In the last column of the table, we quote the results shown in Ref.[29] which were calculated by the lattice simulation.

Table I. The mass spectrum of two-gluon glueballs.

Glueball states (J^{PC})	Mass(GeV)		Mass(MeV)
	Case I	Case II	Lattice results
0^{++}	1.73	2.18	1730(50)(80)
	2.66	3.59	2670(180)(130)
	3.59		
0^{-+}	2.60	2.30	2590(40)(130)
	3.65	3.78	3640(60)(180)
1^{++}	2.73	2.42	
	3.45	3.51	
1^{-+}	2.67	2.59	
	2.87	3.01	
2^{++}	2.43	2.43	2400(25)(120)
2^{-+}	3.32	2.26	3100(30)(150)

As seen from Eq.(4.24), each glueball state is not only assigned by its spin J and parity π, but also related to the mode marked by $(\lambda_1, l_1)(\lambda_2, l_2)$. In this paper, we take low-lying modes to perform the calculation. For the scalar glueballs of quantum numbers $J^{PC} = 0^{++}$ and the tensor ones 2^{++}, according to the angular momentum and the parities of the multipole fields, we take the mixture of the modes $(TE1TE1)$ and $(TM1TM1)$. For the glueball states 0^{-+} and 2^{-+}, the modes are taken to be $(TE1TM1)$ for every glueball, as was similarly done in the investigation within the bag model [19,20]. This means that these glueballs are mainly constructed by the gluons with transverse polarization. But, this does not imply no contribution of the longitudinally polarized gluons to these glueballs. The longitudinal gluons may, as virtual particles, appear in the intermediate states in the effective interaction Hamiltonian. It is emphasized here that for the transverse mode of gluons, as mentioned in Appendix A, the mode $l_i = 0$ is not permitted. This mode can only exist for the longitudinal gluons. Different from the case of massless gluons, the longitudinal mode of massive gluons is possible to take part in formation of some glueballs. For example, the glueball states 1^{-+} can be formed not only by a combination of modes $(M1E1)$ and $(E1L0)$ which gives the states with masses as listed in the table, the modes $(M1E1)$ and $(L1L0)$ can also form the glueball states with masses $3.23 GeV$ and $3.82 GeV$ respectively. For the states 1^{++}, we only take the mode $(L0M1)$ in our calculation because according to the B-S amplitude constructed in Sec.IV (see Eq.(4.21)), the modes $(E1E1), (M1M1)$ are forbidden.

In order to determine the parameter dependence and errors in our calculation, we take three sets of parameters in which we set $\mu = \Lambda_{QCD}$ and γ (in unit GeV^2)$= \Lambda_{QCD}^2$. The results of case I in table I with different parameters (the parameters α_s^0 and N_f remain unchanged) are presented in Table II. We find that the masses of glueballs increase gradually when the set of parameters increase. This indicates that our numerical calculation is stable and the results are reliable.

Table II. The mass spectrum with different parameters.

Glueball states (J^{PC})	Mass(GeV)		
	$\mu = 0.35GeV$	$\mu = 0.45GeV$	$\mu = 0.55GeV$
0^{++}	1.66	1.74	1.81
	2.59	2.68	2.75
	3.52	3.60	3.66
0^{-+}	2.51	2.60	2.68
	3.57	3.66	3.73
1^{++}	2.64	2.73	2.82
	3.37	3.46	3.54
1^{-+}	2.59	2.69	2.78
	2.78	2.88	2.96
2^{++}	2.35	2.45	2.53
2^{-+}	3.25	3.34	3.41

The calculated results show that the gluon mass give an appreciable effect on the glueball masses. In our calculation, the mass of gluon should be around 0.45 GeV. This fact indicates the reasonability of the QCD with massive gluons which is chosen to be the starting point in our calculation. For this kind of QCD, it is necessary to take the longitudinal mode of gluons into account. As shown in Table I, when the longitudinal mode is considered in the intermediate states, the theoretical masses for the states mentioned above would be greatly improved. Otherwise, there would occur a considerable discrepancy between the results given by this paper and the lattice calculation. In addition, as illustrated before, the longitudinal mode allows us to investigate more glueball states which possibly exist in the world.

In comparison with the previous theoretical glueball masses obtained from the B-S equation and the Dirac equation, our results give more support to the lattice predictions [29-31] which are believed to be more reliable because the lattice calculation is based on the QCD first principle and essentially nonperturbative. Within the statistical errors existing in the lattice calculations, our results shown in the first column of Table I can be considered to be consistent with lattice predictions for the low-lying glueballs with masses less than $3.5GeV$, especially, for the lowest scalar glueball state 0^{++} with mass about $1.7GeV$ and the tensor glueball state 2^{++} with mass about $2.4GeV$. The achievement of the better consistence is obviously attributed to the fact that our calculation is fully relativistic and is able to include the contributions arising from the retardation effect and longitudinal mode of the gluon field which could not be considered in the previous investigations [23-25]. Now let us analyze our results in some more detail. It is mentioned that the lowest scalar glueball state 0^{++} and the tensor one 2^{++} have been investigated in many models and the theoretical masses are almost the same even though there are a little difference between different calculations [29-31]. For example, the mass of the lowest state 0^{++} was given by $1754 \pm 65 \pm 86 MeV$ in a recent lattice calculation [31] which is different from that given in Ref.[29]. It is expected that these states may be identified with the pure glueball states and searched out first in future experiments. Aside from the two states mentioned above, the first radial excited state 0^{++} with mass $2.66GeV$ should correspond to the state $2670(180)(130)MeV$ given in the lattice simulation even though whether the latter is a pure glueball or not is still in question [32,33]. The next radial excited state 0^{++} with mass $3.59GeV$ is a new prediction given in this paper which was not predicted in the lattice simulation and the other calculations. As for the pseudoscalar states 0^{-+} and pseudotensor state 2^{-+} are all comparable with the corresponding states presented in the lattice calculation. But, the mass of the state 2^{-+} is little higher than the lattice one. In addition, we note that the states 1^{-+} and 1^{++} were not predicted in the lattice simulation, but the states 1^{-+} with masses $2.67GeV$ and the state 1^{++} with mass $2.73GeV$ are compatible with the recent calculation by the nonrelativistic potential model [18].

In conclusion, it is emphasized that different from some previous investigations, the calculation in this paper is based on the rigorous three-dimensional relativistic equation satisfied by the two-gluon glueball states which is derived from the QCD with massive gluons and represented in the angular momentum representation. Especially, the interaction Hamiltonian in the equation is given a complete expression which provides a firm basis for further study. In this paper, even though we work in the relativistic potential model with introducing phenomenologically a confining potential, the new consideration of the retardation effect and the longitudinal mode of the gluon fields allows us to get the improved theoretical results which are well consistent with the lattice predictions. The only uncertainty in our calculation arises from the introduction of confining potential. Certainly, if a sophisticated confining potential could be found from the exact interaction Hamiltonian derived in this and former papers, it would be anticipated that a relativistic calculation may give more accurate theoretical predictions.

VI. ACKNOWLEDGMENT

VII. APPENDIX A: THE EXPRESSION OF GLUON THREE-LINE VERTEX

This appendix is used to derive the explicit expression of the gluon three-line vertex in the angular momentum representation. As shown in Eqs.(4.7), (4.29) and (4.30) in paper I, the gluon three-line vertex in the interaction Hamiltonian is

$$H_g^3 = -\frac{g}{2} f^{abc} \int d^3x (\overrightarrow{A^{\vec{a}}} \times \overrightarrow{A^{\vec{b}}}) \cdot (\nabla \times \overrightarrow{A^{\vec{c}}}). \tag{A.1}$$

It is represented in the angular momentum representation as follows

$$H_g^3 = \sum_{\alpha_1 \alpha_2 \alpha_3} A(\alpha_1 \alpha_2 \alpha_3) : \mathbf{a}_{\alpha_1} \mathbf{a}_{\alpha_2} \mathbf{a}_{\alpha_3} :, \tag{A.2}$$

where

$$A(\alpha_1 \alpha_2 \alpha_3) = -\frac{g}{2} f^{abc} \int d^3x (\overrightarrow{A}_{\beta_1}^{\lambda_1} \times \overrightarrow{A}_{\beta_2}^{\lambda_2}) \cdot (\nabla \times \overrightarrow{A}_{\beta_3}^{\lambda_3}). \tag{A.3}$$

Here we have set $\alpha_i = (\lambda_i, \beta_i)$ in which $\lambda_i = TE, TM, L$ mark the transverse electric, transverse magnetic and longitudinal modes of the multipole fields and $\beta_i = (l_i, m_i, k_i, \xi_i)$. From now on, we use the symbols $l_i m_i$ to represent the total angular momentum. For later convenience, the relations between the multipole fields which were mentioned in paper I are represented in the matrix form

$$\nabla \times \begin{pmatrix} \overrightarrow{A}_{lm}^{TE}(k\overrightarrow{x}) \\ \overrightarrow{A}_{lm}^{TM}(k\overrightarrow{x}) \\ \overrightarrow{A}_{lm}^{L}(k\overrightarrow{x}) \end{pmatrix} = k \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{A}_{lm}^{TE}(k\overrightarrow{x}) \\ \overrightarrow{A}_{lm}^{TM}(k\overrightarrow{x}) \\ \overrightarrow{A}_{lm}^{L}(k\overrightarrow{x}) \end{pmatrix}. \tag{A.4}$$

When we define

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.5}$$

(A.4) can be rewritten as

$$\nabla \times \overrightarrow{A}_{lm}^{\lambda_i}(k\overrightarrow{x}) = k T_{\lambda_i \lambda_j} \overrightarrow{A}_{lm}^{\lambda_j}(k\overrightarrow{x}). \tag{A.6}$$

Thus, (A.3) reads

$$A(\alpha_1 \alpha_2 \alpha_3) = -\frac{g}{2} f^{abc} k_3 T_{\lambda_3 \lambda_j} \int d^3x (\overrightarrow{A}_{\beta_1}^{\lambda_1} \times \overrightarrow{A}_{\beta_2}^{\lambda_2}) \cdot \overrightarrow{A}_{\beta_3}^{\lambda_j}. \tag{A.7}$$

With the definition

$$\overrightarrow{y}_{lm}^{(\tau)}(k\overrightarrow{x}) = \sqrt{\frac{2}{\pi}} k j_{l+\tau}(kr) \overrightarrow{Y}_{l,l+\tau,m}(\widehat{x}) \equiv \overrightarrow{y}_{ll'm}(k\overrightarrow{x}), \tag{A.8}$$

where $l' = l + \tau$ and $\tau = 0, \pm 1$, the relations shown in Eqs.(3.2)-(3.4) in paper I for the multipole fields can also be written in the matrix form

$$\begin{pmatrix} \overrightarrow{A}_{lm}^{TE}(k\overrightarrow{x}) \\ \overrightarrow{A}_{lm}^{TM}(k\overrightarrow{x}) \\ \overrightarrow{A}_{lm}^{L}(k\overrightarrow{x}) \end{pmatrix} = \begin{pmatrix} i\sqrt{\frac{l+1}{2l+1}} & 0 & -i\sqrt{\frac{l}{2l+1}} \\ 0 & 1 & 0 \\ -i\sqrt{\frac{l}{2l+1}} & 0 & -i\sqrt{\frac{l+1}{2l+1}} \end{pmatrix} \begin{pmatrix} \overrightarrow{y}_{lm}^{(-1)}(k\overrightarrow{x}) \\ \overrightarrow{y}_{lm}^{(0)}(k\overrightarrow{x}) \\ \overrightarrow{y}_{lm}^{(1)}(k\overrightarrow{x}) \end{pmatrix}. \tag{A.9}$$

If we define $(TE, TM, L) = (-1, 0, 1)$ and

$$B(l) = \begin{pmatrix} i\sqrt{\frac{l+1}{2l+1}} & 0 & -i\sqrt{\frac{l}{2l+1}} \\ 0 & 1 & 0 \\ -i\sqrt{\frac{l}{2l+1}} & 0 & -i\sqrt{\frac{l+1}{2l+1}} \end{pmatrix}, \tag{A.10}$$

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where $l \neq 0$ and

$$B(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \tag{A.11}$$

which means only the longitudinal mode survives when $l = 0$, then (A.9) can concisely be written as

$$\overrightarrow{A}_{lm}^{\lambda}(k\overrightarrow{x}) = B(l)_{\lambda\tau}\overrightarrow{y}_{lm}^{(\tau)}(k\overrightarrow{x}). \tag{A.12}$$

After inserting (A.12) into (A.7) and noticing the definition given in Eq.(4.16) in paper I, we have

$$
\begin{aligned}
&A(\alpha_1\alpha_2\alpha_3) \\
&= -\frac{g}{2}f^{abc}k_3\frac{1}{\prod\limits_{i=1}^{3}\sqrt{2\omega(k_i)}}T_{\lambda_3\lambda_j}B^{\xi_1}(l_1)_{\lambda_1\tau_1}B^{\xi_2}(l_2)_{\lambda_2\tau_2}B^{\xi_3}(l_3)_{\lambda_j\tau_3} \\
&\quad \times \int d^3x[\overrightarrow{y}_{l_1l_1'm_1}^{\xi_1}(k_1\overrightarrow{x}) \times \overrightarrow{y}_{l_2l_2'm_2}^{\xi_2}(k_2\overrightarrow{x})] \cdot \overrightarrow{y}_{l_3l_3'm_3}^{\xi_3}(k_3\overrightarrow{x}),
\end{aligned} \tag{A.13}
$$

where we have defined

$$\overrightarrow{y}_{ll'm}^{\xi}(k\overrightarrow{x}) = \begin{cases} \overrightarrow{y}_{ll'm}(k\overrightarrow{x}) & \text{if } \xi = 1 \\ \overrightarrow{y}_{ll'm}^{*}(k\overrightarrow{x}) & \text{if } \xi = -1 \end{cases} \tag{A.14}$$

and

$$B^{\xi}(l_i)_{\lambda\tau} = \begin{cases} B(l_i)_{\lambda\tau} & \text{if } \xi = 1 \\ B^{*}(l_i)_{\lambda\tau} & \text{if } \xi = -1 \end{cases}. \tag{A.15}$$

In light of the the expression in (A.8), the integral over $\overrightarrow{x}$ in (A.13) can be represented as

$$
\begin{aligned}
&\int d^3x[\overrightarrow{y}_{l_1l_1'm_1}^{\xi_1}(k_1\overrightarrow{x}) \times \overrightarrow{y}_{l_2l_2'm_2}^{\xi_2}(k_2\overrightarrow{x})] \cdot \overrightarrow{y}_{l_3l_3'm_3}^{\xi_3}(k_3\overrightarrow{x}) \\
&= (\tfrac{2}{\pi})^{\frac{3}{2}}k_1k_2k_3J_{l_1'l_2'l_3'}(k_1,k_2,k_3)\Gamma(l_i,l_i',m_i,\xi_i),
\end{aligned} \tag{A.16}
$$

where

$$\Gamma(l_i,l_i',m_i,\xi_i) = \int d\Omega(\widehat{x})[\overrightarrow{Y}_{l_1l_1'm_1}^{\xi_1}(\widehat{x}) \times \overrightarrow{Y}_{l_2l_2'm_2}^{\xi_2}(\widehat{x})] \cdot \overrightarrow{Y}_{l_3l_3'm_3}^{\xi_3}(\widehat{x}) \tag{A.17}$$

and

$$J_{l_1'l_2'l_3'}(k_1,k_2,k_3) = \int dr r^2 j_{l_1'}(k_1r)j_{l_2'}(k_2r)j_{l_3'}(k_3r). \tag{A.18}$$

On substituting (A.16) in (A.13), we just give the formula denoted in Eq.(3.22).

First, let us calculate the $\Gamma(l_i,l_i',m_i,\xi_i)$ in the case of $\xi_i = 1$. In this case, we set

$$\Gamma(l_i,l_i',m_i) = \int d\Omega(\widehat{x})[\overrightarrow{Y}_{l_1l_1'm_1}(\widehat{x}) \times \overrightarrow{Y}_{l_2l_2'm_2}(\widehat{x})] \cdot \overrightarrow{Y}_{l_3l_3'm_3}(\widehat{x}). \tag{A.19}$$

By using the following formulas [39,40]:

$$
\begin{aligned}
\overrightarrow{Y}_{ll'm}(\widehat{x}) &= \sum_{m'q}C_{l'm'1q}^{lm}Y_{l'm'}(\widehat{x})\overrightarrow{e}_q, \\
\overrightarrow{e}_{q_1} \times \overrightarrow{e}_{q_2} &= i\sqrt{2}\sum_{s}C_{1q_11q_2}^{1s}\overrightarrow{e}_s, \\
\overrightarrow{e}_q &= (-1)^q\overrightarrow{e}_{-q}^{*}, \quad \overrightarrow{e}_q^{*} \cdot \overrightarrow{e}_{q'} = \delta_{qq'},
\end{aligned} \tag{A.20}
$$

we find

$$
\begin{aligned}
\Gamma(l_i,l_i',m_i) &= i\sqrt{2}\sum_{n_1n_2n_3q_1q_2q_3}(-1)^{q_3}C_{l_1'n_11q_1}^{l_1m_1}C_{l_2'n_21q_2}^{l_2m_2}C_{l_3'n_31q_3}^{l_3m_3}C_{1q_11q_2}^{1,-q_3} \\
&\quad \times \int d\Omega(\widehat{x})Y_{l_1'n_1}(\widehat{x})Y_{l_2'n_2}(\widehat{x})Y_{l_3'n_3}(\widehat{x}).
\end{aligned} \tag{A.21}
$$

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Employing the familiar formula for the above integral and the definition and property of 3-j and 9-j symbols for the angular momentum couplings [39,40], one can get

$$\Gamma(l_i, l_i', m_i) = i\left[\frac{(2l_1+1)(2l_2+1)(2l_3+1)(2l_1'+1)(2l_2'+1)(2l_3'+1)}{2\pi}\right]^{\frac{1}{2}}$$
$$\times \begin{pmatrix} l_1' & l_2' & l_3' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ l_1' & l_2' & l_3' \\ l_1 & l_2 & l_3 \end{Bmatrix}. \tag{A.22}$$

In the case of $\xi = -1$, it is easy to find

$$\overrightarrow{Y}^*_{ll'm}(\widehat{x}) = (-1)^{l+l'+m+1}\overrightarrow{Y}_{ll'-m}(\widehat{x}). \tag{A.23}$$

If we define

$$\overrightarrow{Y}^{\eta}_{ll'm}(\widehat{x}) = \begin{cases} \overrightarrow{Y}_{ll'm}(\widehat{x}) & \text{if } \xi = 1 \\ \overrightarrow{Y}^*_{ll'm}(\widehat{x})) & \text{if } \xi = -1 \end{cases}, \tag{A.24}$$

the vector spherical harmonics $\overrightarrow{Y}_{ll'm}(\widehat{x})$ and $\overrightarrow{Y}^*_{ll'm}(\widehat{x})$ can be represented in an unified form

$$\overrightarrow{Y}^{\eta}_{ll'm}(\widehat{x}) = (-1)^{(l+l'+m+1)\sin[\frac{(1-\eta)\pi}{4}]}\overrightarrow{Y}_{ll',\eta m}(\widehat{x}). \tag{A.25}$$

Thus, according to (A.22)- (A.25), we have

$$\Gamma(l_i, l_i', m_i, \eta_i) \equiv \int d\Omega(\widehat{x})[\overrightarrow{Y}^{\eta_1}_{l_1 l_1' m_1}(\widehat{x}) \times \overrightarrow{Y}^{\eta_2}_{l_2 l_2' m_2}(\widehat{x})] \cdot \overrightarrow{Y}^{\eta_3}_{l_3 l_3' m_3}(\widehat{x})$$
$$= \prod_{i=1}^{3}(-1)^{(l_i+l_i'+m_i+1)\sin[\frac{(1-\eta_i)\pi}{4}]}\Gamma(l_i, l_i', \eta_i m_i), \tag{A.26}$$

where $\Gamma(l_i, l_i', \eta_i m_i)$, as expressed in Eq.(3.24), is directly written out from (A.22) with replacing m_i by $\eta_i m_i$.

Let us turn to compute the integral in (A.18) following the method proposed by one of the authors in this paper and his coworker in their early publications [41]. As we know, there is a momentum conservation in the gluon three-line vertex: $\overrightarrow{k_1} + \overrightarrow{k_2} + \overrightarrow{k_3} = 0$ which gives a certain restriction on the magnitudes of the three momenta. In fact, from $k_1^2 = (\overrightarrow{k_2} + \overrightarrow{k_3})^2 = k_2^2 + k_3^2 + 2k_2 k_3 \cos\theta_{12}$, it is seen that when $\cos\theta_{12} = \pm 1$, we have $k_1 = k_2 + k_3$ and $k_1 = |k_2 - k_3|$. This implies that only when the conditions $k_1 + k_2 \geqslant k_3, k_2 + k_3 \geqslant k_1$ and $k_1 + k_3 \geqslant k_2$ or

$$k_1 + k_2 - k_3 \geqslant 0, k_2 + k_3 - k_1 \geqslant 0, k_1 + k_3 - k_2 \geqslant 0 \tag{A.27}$$

are simultaneously satisfied, the momentum conservation holds; whereas, when $k_1 > k_2 + k_3$, the momentum conservation is violated. In addition, adding any two inequalities in (A.27) together, we find $k_i \geqslant 0$, therefore, each k_i varies from zero to infinity. In later derivations, the following relations are useful

$$h_l^{(1)}(x) = j_l(x) + in_l(x), \ h_l^{(2)}(x) = j_l(x) - in_l(x),$$
$$j_l(-x) = (-1)^l j_l(x), \quad h_l^{(1)}(-x) = (-1)^l h_l^{(2)}(x), \tag{A.28}$$

where $j_l(x)$ is the spherical Bessel function, $n_l(x)$ the spherical Neumann function, $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$ are the first class spherical Hankel function and the second class one respectively. The asymptotic behaviors of these functions are as follows. When $x \to 0$,

$$j_l(x) \longrightarrow \frac{2^l l! x^l}{(2l+1)!}, \qquad n_l(x) \longrightarrow \frac{-(2l-1)!!}{x^{l+1}},$$
$$h_l^{(1)}(x) \longrightarrow \frac{-i(2l)!}{2^l l! x^{l+1}}, \qquad h_l^{(2)}(x) \longrightarrow \frac{i(2l-1)!!}{x^{l+1}}, \tag{A.29}$$

and when $x \to \infty$,

$$j_l(x) \longrightarrow \frac{-i}{2x}[e^{i(x-\frac{l\pi}{2})} - e^{-i(x-\frac{l\pi}{2})}],$$
$$h_l^{(1)}(x) \longrightarrow \frac{-i}{x}e^{i(x-\frac{l\pi}{2})}, h_l^{(2)}(x) \longrightarrow \frac{i}{x}e^{-i(x-\frac{l\pi}{2})}. \tag{A.30}$$

First, we prove that the integral $J_{l_1' l_2' l_3'}(k_1, k_2, k_3)$ vanishes in the case of $k_1 > k_2 + k_3$. In this case, considering the analytical property of the functions $j_l(x)$ and $h_l^{(1)}(x)$ as shown in (A.29) and (A.30), the following integral along the contour C on the upper half complex plane of r as depicted in Fig.1 is zero

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$$\oint_C dr r^2 h_{l_1}^{(1)}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) = 0. \tag{A.31}$$

The contour C can be divided into four parts, $C = C_0 + (-\infty, 0^-) + C_1 + (0^+, +\infty)$. Clearly, the integral along the large half circle C_1 vanishes when $|r|$ tends to infinity. Thus, noticing the relations in (A.28), $l_i \geqslant 0$ and $l_1 + l_2 + l_3 = even$ which is implied by the first 3-j symbol in (A.22), one can get from (A.31)

$$J_{l'_1 l'_2 l'_3}(k_1, k_2, k_3) = -\frac{1}{2} \int_{C_0} dr r^2 h_{l_1}^{(1)}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r). \tag{A.32}$$

Substituting the series expansions

$$
\begin{aligned}
j_l(kr) &= \frac{\sqrt{\pi}}{2} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu (\frac{kr}{2})^{2\mu+l}}{\Gamma(\mu+1)\Gamma(\mu+l+\frac{3}{2})}, \\
n_l(kr) &= \frac{\sqrt{\pi}}{2} \sum_{\mu=0}^{\infty} \frac{(-1)^{l+\mu+1}(\frac{kr}{2})^{2\mu-l-1}}{\Gamma(\mu+1)\Gamma(\mu-l+\frac{1}{2})}
\end{aligned}
\tag{A.33}
$$

into the right hand side of (A.32), it is easy to find that the integral along the circle C_0 around the origin also vanishes when $|r|$ goes to zero. Thus, we reach the following result

$$J_{l'_1 l'_2 l'_3}(k_1, k_2, k_3) = 0. \tag{A.34}$$

Next, we compute the integral under the conditions shown in (A.27). In view of these conditions and the asymptotic behaviors of $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$ shown in (A.30), the function $f(r)$ defined by

$$
\begin{aligned}
f(r) &= h_{l_1}^{(1)}(k_1 r) h_{l_2}^{(1)}(k_2 r) h_{l_3}^{(2)}(k_3 r) + h_{l_1}^{(1)}(k_1 r) h_{l_2}^{(2)}(k_2 r) h_{l_3}^{(1)}(k_3 r) \\
&+ h_{l_1}^{(2)}(k_1 r) h_{l_2}^{(1)}(k_2 r) h_{l_3}^{(1)}(k_3 r) + h_{l_1}^{(1)}(k_1 r) h_{l_2}^{(1)}(k_2 r) h_{l_3}^{(1)}(k_3 r)
\end{aligned}
\tag{A.35}
$$

is analytical on the upper half complex plane of r except for at the origin. Therefore, we have

$$\int_C dr r^2 f(r) = 0, \tag{A.36}$$

where the contour C is still represented in Fig.1. Due to the conditions in (A.27), the integral along C_1 still vanishes. Thus, from the above integral, we get

$$
\begin{aligned}
J_{l'_1 l'_2 l'_3}(k_1, k_2, k_3) &= \frac{1}{8}\left\{ \int_0^\infty dr r^2 f(r) + \int_{-\infty}^0 dr r^2 f(r) \right\} \\
&= -\frac{1}{8} \int_{C_0} dr r^2 f(r).
\end{aligned}
\tag{A.37}
$$

In accordance with (A.28), the function $f(r)$ can be written as

$$
\begin{aligned}
&f(r) \\
&= 4 j_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) + 2 i n_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) \\
&+ 2 i j_{l_1}(k_1 r) n_{l_2}(k_2 r) j_{l_3}(k_3 r) + 2 i j_{l_1}(k_1 r) j_{l_2}(k_2 r) n_{l_3}(k_3 r) \\
&+ 2 i n_{l_1}(k_1 r) n_{l_2}(k_2 r) n_{l_3}(k_3 r).
\end{aligned}
\tag{A.38}
$$

Inserting this expression into (A.37) and using the series representation in (A.33), it can be found that except for the last term, the other terms in (A.38) all give no contribution to the integral. Therefore, we have

$$
\begin{aligned}
J_{l'_1 l'_2 l'_3}(k_1, k_2, k_3) &= -\frac{i}{4} \int_{C_0} dr r^2 n_{l_1}(k_1 r) n_{l_2}(k_2 r) n_{l_3}(k_3 r) \\
&= \frac{i\pi^{\frac{3}{2}}}{32} \sum_{\mu_1, \mu_2, \mu_3} \prod_{i=1}^3 \frac{(-1)^{\mu_i+l_i}(\frac{k_i}{2})^{2\mu_i-l_i-1}}{\Gamma(\mu_i+1)\Gamma(\mu_i-l_i+\frac{1}{2})} \int_{C_0} dr r^{2(\mu_1+\mu_2+\mu_3)-l_1-l_2-l_3-1}.
\end{aligned}
\tag{A.39}
$$

Setting $r = \rho e^{i\theta}$ and noticing $2(\mu_1 + \mu_2 + \mu_3) - l_1 - l_2 - l_3 = even$ and $\int_\pi^0 d\theta e^{i2m\theta} = -\pi \delta_{m,0}$ here m is an integer, we finally obtain the expression as shown in Eq.(3.23).

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VIII. APPENDIX B: THE EXPRESSION OF GLUON FOUR-LINE VERTEX

In this appendix we would like to derive the explicit expression of the gluon four-line vertex in the angular momentum representation for completeness although the vertex gives no contribution to the equation (2.27) in the lowest order approximation due to $S_2(\rho\sigma\tau\lambda, \gamma\delta) = 0$ as shown in Eq.(3.1). The four-line vertex in the interaction Hamiltonian at $t = 0$ which was described in Eqs.(4.7), (4.29) and (4.31) in paper I may be rewritten as

$$
\begin{aligned}
H_g^4 &= \frac{g^2}{4} f^{abe} f^{cde} \int d^3x (\overrightarrow{A^{\vec{a}}} \cdot \overrightarrow{A^{\vec{c}}})(\overrightarrow{A^{\vec{b}}} \cdot \overrightarrow{A^{\vec{d}}}) \\
&= \sum_{\alpha\beta\gamma\delta} B(\alpha_1\alpha_2\alpha_3\alpha_4) : \mathbf{a}_{\alpha_1}\mathbf{a}_{\alpha_2}\mathbf{a}_{\alpha_3}\mathbf{a}_{\alpha_4} :,
\end{aligned}
\tag{B.1}
$$

where the second equality is obtained by substituting the expansion of gluon fields in terms of the multipole fields into the first equality and the coefficient function is of the form

$$
B(\alpha_1\alpha_2\alpha_3\alpha_4) = \frac{g^2}{4} f^{abe} f^{cde} \int d^3x (\overrightarrow{A}_{\alpha_1}^{\lambda_1} \cdot \overrightarrow{A}_{\alpha_2}^{\lambda_2})(\overrightarrow{A}_{\alpha_3}^{\lambda_3} \cdot \overrightarrow{A}_{\alpha_4}^{\lambda_4}).
\tag{B.2}
$$

By making use of the representation in (A.8)-(A.12), (A.14) and (A.15) and noticing the definition given in Eq.(4.16) in paper I, the $B(\alpha_1\alpha_2\alpha_3\alpha_4)$ can be represented as

$$
\begin{aligned}
B(\alpha_1\alpha_2\alpha_3\alpha_4) &= \frac{g^2}{\pi^2} f^{abe} f^{cde} \prod_{i=1}^{4} \frac{k_i}{\sqrt{2\omega(k_i)}} B^{\xi_i}(l_i)_{\lambda_i\tau_i} \\
&\times J_{l_1' l_2' l_3' l_4'}(k_1, k_2, k_3, k_4) \widetilde{\Gamma}(l_i, l_i', m_i, \eta_i),
\end{aligned}
\tag{B.3}
$$

where

$$
J_{l_1' l_2' l_3' l_4'}(k_1, k_2, k_3, k_4) = \int dr r^2 j_{l_1'}(k_1 r) j_{l_2'}(k_2 r) j_{l_3'}(k_3 r) j_{l_4'}(k_4 r)
\tag{B.4}
$$

and

$$
\widetilde{\Gamma}(l_i, l_i', m_i, \eta_i) = \int d\Omega(\hat{x}) \overrightarrow{Y}_{l_1 l_1' m_1}^{\eta_1}(\hat{x}) \cdot \overrightarrow{Y}_{l_2 l_2' m_2}^{\eta_2}(\hat{x}) \overrightarrow{Y}_{l_3 l_3' m_3}^{\eta_3}(\hat{x}) \cdot \overrightarrow{Y}_{l_4 l_4' m_4}^{\eta_4}(\hat{x}),
\tag{B.5}
$$

here the notation in (A.24) has been used. Inserting (A.20) and (A.25) into (B.5) and employing the familiar formulas for the integrals of spherical harmonics and for the angular momentum coupling [39,40], it is not difficult to get

$$
\begin{aligned}
&\widetilde{\Gamma}(l_i, l_i', m_i, \eta_i) \\
&= \frac{1}{4\pi} \sum_{lm} (-1)^{l_1+l_2+l_3'+l_4'+m} (2l+1) \prod_{i=1}^{4} (-1)^{(l_i+l_i'+m_i+1)} \sin[\frac{(1-\eta_i)\pi}{4}] \\
&\times [(2l+1)(2l'+1)]^{\frac{1}{2}} \begin{pmatrix} l_1' & l_3' & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_2' & l_4' & l \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} l_1 & l_3 & l \\ l_3' & l_1' & 1 \end{Bmatrix} \\
&\times \begin{Bmatrix} l_2 & l_4 & l \\ l_4' & l_2' & 1 \end{Bmatrix} \begin{pmatrix} l_1 & l_3 & l \\ \eta_1 m_1 & \eta_3 m_3 & m \end{pmatrix} \begin{pmatrix} l_2 & l_4 & l \\ \eta_2 m_2 & \eta_4 m_4 & -m \end{pmatrix}.
\end{aligned}
\tag{B.6}
$$

To compute the integral in (B.4), we first examine the conditions satisfied by the magnitudes of the momenta. From the momentum conservation $\overrightarrow{k_1} + \overrightarrow{k_2} + \overrightarrow{k_3} + \overrightarrow{k_4} = 0$, we have

$$
\begin{aligned}
k_1^2 &= (\overrightarrow{k_2} + \overrightarrow{k_3} + \overrightarrow{k_4})^2 = k_2^2 + k_3^2 + k_4^2 \\
&+ 2k_2 k_3 \cos\theta_{23} + 2k_2 k_4 \cos\theta_{24} + 2k_3 k_4 \cos\theta_{34}.
\end{aligned}
\tag{B.7}
$$

From the above equality, we may find the maximum and minimum of k_1 by setting $\theta_{ij} = 0$ or π. There are only four ways which permit us to take the values of $\theta_{ij} = 0$ or π. These are (1) $\theta_{23} = \theta_{24} = \theta_{34} = 0$; (2) $\theta_{23} = \theta_{24} = \pi, \theta_{34} = 0$; (3) $\theta_{23} = \theta_{34} = \pi, \theta_{24} = 0$; (4) $\theta_{24} = \theta_{34} = \pi, \theta_{23} = 0$. Correspondingly, we get from (B.7) the equalities $k_1 = k_2 + k_3 + k_4, k_1 = |k_2 - k_3 - k_4|, k_1 = |k_3 - k_2 - k_4|$ and $k_1 = |k_4 - k_2 - k_3|$. From these equalities we find the following restriction conditions which are consistent with the momentum conservation

$$
\begin{aligned}
&k_2 + k_3 + k_4 - k_1 \geqslant 0, k_1 + k_3 + k_4 - k_2 \geqslant 0, \\
&k_1 + k_2 + k_4 - k_3 \geqslant 0, k_1 + k_2 + k_3 - k_4 \geqslant 0, \\
&k_1 + k_2 - k_3 - k_4 \geqslant 0, k_1 + k_3 - k_2 - k_4 \geqslant 0, \\
&k_1 + k_4 - k_2 - k_3 \geqslant 0, k_1 + k_2 + k_3 + k_4 \geqslant 0.
\end{aligned}
\tag{B.8}
$$

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By adding some two of the above inequalities, one may see $k_i \geqslant 0$. And similar to the proof described in the preceding appendix, it can be proved that the integral in (B.4) is merely nonvanishing provided that the conditions in (B.8) are respected. According to the above conditions, it is obvious that the function defined below is analytical on upper half complex plane of r

$$
\begin{aligned}
&F(r) \\
&= h_{l_1}^{(1)}(k_1 r) h_{l_2}^{(1)}(k_2 r) h_{l_3}^{(1)}(k_3 r) h_{l_4}^{(2)}(k_4 r) + h_{l_1}^{(1)}(k_1 r) h_{l_2}^{(1)}(k_2 r) h_{l_3}^{(2)}(k_3 r) h_{l_4}^{(1)}(k_4 r) \\
&+ h_{l_1}^{(1)}(k_1 r) h_{l_2}^{(2)}(k_2 r) h_{l_3}^{(1)}(k_3 r) h_{l_4}^{(1)}(k_4 r) + h_{l_1}^{(2)}(k_1 r) h_{l_2}^{(1)}(k_2 r) h_{l_3}^{(1)}(k_3 r) h_{l_4}^{(1)}(k_4 r) \\
&+ h_{l_1}^{(1)}(k_1 r) h_{l_2}^{(1)}(k_2 r) h_{l_3}^{(2)}(k_3 r) h_{l_4}^{(2)}(k_4 r) + h_{l_1}^{(1)}(k_1 r) h_{l_2}^{(2)}(k_2 r) h_{l_3}^{(1)}(k_3 r) h_{l_4}^{(2)}(k_4 r) \\
&+ h_{l_1}^{(1)}(k_1 r) h_{l_2}^{(2)}(k_2 r) h_{l_3}^{(2)}(k_3 r) h_{l_4}^{(1)}(k_4 r + h_{l_1}^{(1)}(k_1 r) h_{l_2}^{(1)}(k_2 r) h_{l_3}^{(1)}(k_3 r) h_{l_4}^{(1)}(k_4 r)
\end{aligned}
\tag{B.9}
$$

where the l_i' have been replaced by l_i for convenience. Therefore, based on the Cauchy theorem, the integral along the contour C as depicted in Fig.1 is zero,

$$
\int_C dr r^2 F(r) = 0
\tag{B.10}
$$

From this equation, noticing $l_1 + l_2 + l_3 + l_4 = even$ as implied by the first two 3-j symbols in (B.6), we obtain

$$
\begin{aligned}
J_{l_1' l_2' l_3' l_4}(k_1, k_2, k_3, k_4) &= \tfrac{1}{16}\{\int_0^\infty dr r^2 F(r) + \int_{-\infty}^0 dr r^2 F(r)\} \\
&= -\tfrac{1}{16}\int_{C_0} dr r^2 F(r).
\end{aligned}
\tag{B.11}
$$

In view of the relations in (A.28), the function $F(r)$ can be represented as

$$
\begin{aligned}
&F(r) \\
&= 8 j_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) j_{l_4}(k_4 r) + 2 j_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) n_{l_4}(k_4 r) \\
&+ 2 j_{l_1}(k_1 r) j_{l_2}(k_2 r) n_{l_3}(k_3 r) j_{l_4}(k_4 r) + 2 i j_{l_1}(k_1 r) n_{l_2}(k_2 r) j_{l_3}(k_3 r) j_{l_4}(k_4 r) \\
&+ 6 i n_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) j_{l_4}(k_4 r) + 2 i n_{l_1}(k_1 r) j_{l_2}(k_2 r) n_{l_3}(k_3 r) n_{l_4}(k_4 r) \\
&+ 2 i n_{l_1}(k_1 r) n_{l_2}(k_2 r) j_{l_3}(k_3 r) n_{l_4}(k_4 r) + 2 i n_{l_1}(k_1 r) n_{l_2}(k_2 r) n_{l_3}(k_3 r) j_{l_4}(k_4 r) \\
&- 2 i j_{l_1}(k_1 r) n_{l_2}(k_2 r) n_{l_3}(k_3 r) n_{l_4}(k_4).
\end{aligned}
\tag{B.12}
$$

Upon inserting the above expression into (B.11), using the series representation in (A.33) and considering the relations among the angular momenta which are implied by the 3-j and 9-j symbols in (B.6), it is easily verified that the first five terms in (B.12) give vanishing contributions to the integral. The nonvanishing contributions given by the last four terms can be calculated by the same procedure as described in the last part of Appendix A. The result is

$$
\begin{aligned}
&J_{l_1' l_2' l_3' l_4}(k_1, k_2, k_3, k_4) \\
&= \frac{\pi^3}{16}(-)^{\frac{1}{2}(l_1+l_2+l_3+l_4)} \sum_{\mu_1,\mu_2,\mu_3,\mu_4=0}^{\infty} \delta_{2(\mu_1+\mu_2+\mu_3+\mu_4),l_2'+l_3'+l_4'-l_1'} \\
&\times \sum_{i=1}^{4} \frac{(1-2\delta_{i1})}{\Gamma(\mu_i+1)\Gamma(\mu_i+l_i+\frac{3}{2})} \prod_{i=1(j\neq i)}^{4} \frac{(k_j/k_1)^{2\mu_j-l_j}}{k_j \Gamma(\mu_j+1)\Gamma(\mu_j-l_j+\frac{1}{2})}.
\end{aligned}
\tag{B.13}
$$

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X. FIGURE CAPTION

Fig.1. The contour for the integrals containing three and four spherical Bessel functions.